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PREFACE

This paper presents the results of a study of imposing a three-dimensional perturbation on a barotropic westerly current. The purpose of the present investigation was to present theoretical solutions to the following basic problems: (1) the determination of the three dimensional structure of the wave; (2) the determination of the speed of propagation of the wave; (3) the value of considering vertical motion in the wave motion.

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TABLE OF SYMBOLS AND ABBREVIATIONS

u	horizontal velocity perturbation along x-axis
v	horizontal velocity perturbation along y-axis
w	vertical velocity perturbation along z-axis
p	perturbation component of pressure
c	wave velocity
U	constant zonal velocity along x-axis
f	($= 2\Omega \sin\varphi$); the z component of coriolis acceleration
β	($= 2\Omega \cos\varphi$); the y component of coriolis acceleration
D	proportionality constant between u and v
$H(z);F(z);G(z)$ and their derivatives	functions of the vertical perturbation
ρ	density of atmosphere
* h_e	($= \frac{RT_e}{g}$); height of homogeneous layer of temperature T_e
t	time, counted zero when wave passes origin of cartesian coordinates system
Γ	vertical lapse rate of temperature
* T_e	absolute temperature
φ	angle of latitude
g	acceleration of gravity
E	mean radius of the earth
N	($= 2\pi E \cos\varphi / L$) wave number
$\epsilon = 1 - \frac{R}{g}$	
$M\left(= \left\vert\frac{u_{max}}{w_{max}}\right\vert\right)$	ratio of absolute value of maximum horizontal perturbation velocity to that of maximum vertical perturbation velocity

h_T	height of the tropopause	
K	$= \Gamma/T_0 (g/R\Gamma - 1)^{-1}$	; a parameter
α	$= \Gamma/T_0$	; a parameter
m	$= 2\pi/L$	; a parameter
A	$= \beta K D m / f(D^2-1)$	; a parameter
B	$= -D^2 m^2 / D^2 - 1$	; a parameter
S_e	$= 1/2h_e$	; a parameter
a	$= -\beta^2 K [1 - fDm/\beta K]^2 / f^2 \alpha (D^2-1)$	; a parameter
b	$= K[1 - fDm/\beta K]^2/\alpha$	; a parameter
r	$= -fDm(1 - fDm/\beta K)/\beta\alpha$	; a parameter
Z	$= 2mh_2$	; a parameter in the stratosphere
Z_1	$= 2mh_1/\epsilon$	; a parameter in the troposphere
* Y_e	$= Z[n_e^2 - 1]^{\frac{1}{2}}$	; an equality with appropriate Z
C_n	[for $n = 1, 2, 3 \cdots$]	; constants
* n_e	Discriminant used in evaluation of D	

* e = 0 denotes quantity at the surface
e = 1 denotes tropospheric quantity
e = 2 denotes stratospheric quantity
e = 3 denotes mean tropospheric quantity

I. INTRODUCTION

The theoretical study of the behavior of large-scale flow patterns in the extratropical zones of the earth, associated with migratory long waves in the belt of prevailing westerlies, occupies an important place in meteorological literature.

Many informative theoretical and synoptic studies of the behavior of these large-scale flow patterns have been made since V. Bjerknes [5], in 1916, advanced a theory, based on general hydrodynamic considerations, that cyclones originate as dynamically unstable wavelike disturbances in the westerly current. In 1923, H. H. Clayton [27] made the first systematic use of zonal index by means of the trend of mean pressure versus latitude to determine increasing and decreasing zonal circulation. By 1933, V. Bjerknes and collaborators [27] had computed the zonal velocity distribution from zonal temperature and pressure distribution at the ground, showing the maximum horizontal velocity normally occurs just below the tropopause and concluded that there were 2-3 waves around the 60° latitude circle and slightly above 4 around the 30° circle. In 1937, J. Bjerknes [3,28] offered a simple explanation for the displacement of perturbations superimposed on zonal distribution. Variations in latitude and pressure perturbations, with simultaneous transport of air, result in changes of amplitude or longitudinal displacement of long waves. This result was obtained by combining the equations of motion with the equation of continuity. Rossby [28] pointed out two

basic objections, first, the omission of acceleration terms which may be of the same order of magnitude as the centrifugal and deflecting force, and second, the displacement of observed flow patterns caused by isallobaric systems resulting from divergence associated with initial wind distribution. Rossby [27] extended the J. Bjerknes paper reasoning that since the depth of the atmosphere was much smaller than the horizontal scale of large disturbances, it was permissible to treat the atmospheric motion as two-dimensional, non-divergent flow. Further, he deduced that between the longer westerly moving waves, there existed an intermediate wave length, such that its perturbation remained stationary. The speed of propagation of moving waves is dependent on the strength of the westerlies and the wave length, and was based on the consideration of a change in vorticity in a vertical air column which has been displaced from one latitude to another. Haurwitz [16, 17] extended Rossby's 1939 solution by studying the effects of lateral limits of the wave and a variable Coriolis parameter on a curved earth.

Namias and Clapp [22] investigated Rossby's wave speed formula statistically. Their verification was generally successful and they concluded that amplitude and zonal velocity were both important in determining the stationary wave length.

V. Bjerknes had, by this time, found the number of waves around the 45° latitude circle to be between four and six. In 1944, J. Bjerknes and Holmboe [19] derived a formula analagous to that of Rossby for the more general barotropic atmosphere which agreed qualitatively with that of J. Bjerknes' of 1937.

By 1947, Charney [5] in his baroclinic atmosphere theory, based on J. Bjerknes and Holmboe's theory of wave motion in a baroclinic atmosphere, introduced a solution to the barotropic wave. With this theory, he obtained a solution, based on the principle that a wave will travel with such a speed that the pressure tendencies arise from the displacement of the pressure pattern in accordance with the field of horizontal divergence. Charney found this physical explanation acceptable, but the dynamics of the wave could not be solved by analytic methods based on semi-empirical considerations of gradient wind. Charney [1947] integrated the equations of motion, assuming motion to be adiabatic, and thus determined the speed of propagation, stability criteria and the three-dimensional wave structure. In 1948, Eliassen [12] generalized the Rossby-Holmboe wave formula, but he neglected vertical acceleration and the Coriolis parameter when it appeared in multiplicative combination with the vertical velocity. In 1949, Charney and Eliassen [7] extended numerical analysis by use of the quasi-geostrophic approximation.

In a field where horizontal divergence was so small that the absolute vorticity of air parcels was conserved and the scale of motions was large enough that the Coriolis parameter variation with latitude was important in determining wave dimension, Cressman [9] scrutinized the Rossby wave velocity formula and found retrogression in a series of quasi-stationary long waves began when the basic zonal velocity decreased in speed or shifted south. Such retrogression increased the long wave number (i.e., formed a new cold trough).

Continuing his previous investigation, Charney [7, 8] found that the mean flow with respect to pressure could be described as two-dimensional non-divergent in an equivalent barotropic atmosphere. He used this method as a basis for partially successful twenty-four hour forecasts of the 500 mb flow pattern by numerical integration of the non-linear vorticity equation. The simplified equations derived expressed conservation of the vertical component of absolute potential vorticity. The hydrostatic equation together with the individual rate of change of vorticity evaluated geostrophically, reduced to a single partial differential equation for pressure tendency. The basis of correspondence between the barotropic and baroclinic atmosphere lay in the notion of an equivalent barotropic atmosphere, in which horizontal motion at a particular level approximated the actual motion.

It can readily be seen that a great amount of effort has been placed on the long wave theory in an effort to gain a forecasting tool. The use of vertical motions has suffered in comparison, but primarily because of the sparsity of observations and the necessity to rely upon indirect information. Investigations of the vertical component of motion were undertaken as far back as 1911, when V. Bjerknes [13] described a method for such computations. In 1913, Exner [13] suggested the high correlation between 9 km pressure and temperature could be accounted for by vertical motions. Hesselberg,

in 1915, extended Exner's argument by showing that the presence of persistent vertical motions of 1-10 centimeters per second would account for the observed magnitude of pressure-temperature changes. Jeffreys [24] in his 1922 study of vertical velocities necessary to maintain hailstones, found vertical velocities of any magnitude at all occurred only in eddies and could be ignored in treating mean motion. In 1927, Haurwitz [13] showed that in an atmosphere of constant lapse rate, that the maximum vertical component of velocity was in the region of maximum divergence and was of the order of eight centimeters per second. Brunt, [4] in his Memoirs [1934], found vertical velocities involving divergence were of the order of a few centimeters per second.

As observational information increased, quantitative analysis of divergence and vertical currents became more practicable. Durst [11], in 1940, found vertical velocities of a few centimeters per second; Fleagle [13], in 1945, Gaviola and Fuertes [15], in 1947, assumed adiabatic processes and computed the vertical component of velocity. Panofsky [23], in 1946, from third degree polynomials, found values of the same magnitude (i.e., a few centimeters per second).

While vertical motions of several meters per second have been observed, their duration was short and smoothed out over a twelve-hour period. Thus, the high valued measurements by theodolites as observed by Süring [15] have not altered the basic contention that persistent vertical motions are of a much smaller order.

Charney [6], in assigning magnitudes to vertical motions in 1948, explained that the failure of the tendency equation in pressure prognostication was due to the discounting of vertical motion. Charney [8], further determined that forecasts with the quasi-geostrophic assumption could be achieved only when the distribution of vertical motion and sea-level pressure distribution were known. Thus, it is necessary to extend numerical forecasts to permit forecasting vertical motion. In 1949, Bellamy [2] described a method of objective calculation of vertical velocities. In 1951, Phillips [25] indicated that the equivalent barotropic atmosphere was incapable of providing distribution of vertical motion and sea-level pressure adequately for forecasting the "weather".

In view of the interest in vertical velocity distribution, it is the purpose of the present investigation to present a theoretical solution of a two-layer barotropic, incompressible atmosphere with three-dimensional velocity divergence zero, introducing vertical motion as a perturbation.

II. FORMULATION AND RESULTS OF THE PROBLEM

A brief summary of the principal subject matter of the investigation unhampered by mathematical detail is now given to set forth the procedure followed and results obtained.

Chapter III concerns the construction of a barotropic model. The troposphere is characterized by a constant vertical lapse rate; the stratosphere is assumed to be isothermal, with constant zonal wind throughout the atmosphere. The equations of motion are given in ordinary cartesian coordinates for an incompressible atmosphere. Horizontal velocity divergence is assumed to be compensated by vertical motion. Boundary conditions are formulated. It is shown that the model is consistent with the boundary conditions and the steady state motions prescribed.

The last section of Chapter III, treating the actual motion as a small perturbation superimposed on an undisturbed constant zonal current, is now discussed. The velocity components, as well as the pressure component, are expressed as simple harmonic perturbations of infinite lateral extent traveling in a west-east direction at constant velocity. These perturbation components are functions of height as well as of horizontal coordinates and time. The vertical velocity, involving a general expression of the z-coordinate, is used as a basis for the determination of all other velocity components. To simplify the mathematical difficulties, the meridional perturbation amplitude is

made proportional to that of the zonal perturbation. The zonal amplitude, in turn, by the prerequisite of zero of velocity divergence, is simply related to the vertical motion amplitude function, H(z). It is shown that pressure is also a function of the vertical motion amplitude function.

Since the vertical amplitude is in terms of H(z), the amplitudes of the other components are determinable from H(z). The determination of the proportionality factor between the meridional and latitudinal perturbation velocities assumes an important role in the solution of the problem. Upon its determination rests the dependency of the wave velocity and the maximum allowable meridional perturbation velocity for a particular maximum zonal perturbation velocity. By analogy to Rossby's wave velocity, a restriction that the proportionality factor be less than unity is assumed. It is shown that the proportionality factor is dependent upon the discriminant of an auxiliary quadratic equation. Following the boundary conditions this discriminant must have a value slightly in excess of unity. The discriminant value must be determined from an inequality. The choice of the value of this inequality plays a dominant role in any further solution of this problem.

The problem then becomes one of determining the speed of propagation of the wave and the character of the vertical perturbation which introduces the amplitude as a function of height. The determination of these are dependent upon wave length and the parameters

characterizing the mean state of the atmosphere, namely, the height of the tropopause, the vertical velocity at the tropopause, the zonal wind, the vertical lapse rate, the mean latitude and surface density, pressure, and temperature.

In Chapter IV, the form of the perturbation is set forth. The perturbation functions are solved in the stratosphere, and the troposphere (by approximation). The proportionality constant between meridional and zonal perturbation velocities is determined. This constant is of great importance in the determination of the wave speed and stationary wave length. Amplitude evaluation is determined. The complete solution of the approximate case is shown diagramatically.

In Chapter V, the tropospheric perturbation functions are solved without simplifying the coefficients of the differential equation satisfied by the amplitude function, H(z).

Due to barotropy, no phase change is introduced into the structure of the wave. Vertical cells do appear in the perturbation velocity structure moving with the basic wave velocity. The perturbation fields of velocity diminish with increasing height and eventually approach zero. The wave in the meridional velocity field lags 90° behind the wave in the pressure field and the wave in the density field is in phase with the pressure wave. The zonal perturbation is in phase with the pressure wave, whereas the vertical velocity perturbation leads the pressure wave by 90°. These phase relationships are shown in Figure 3.

III. THE MATHEMATICAL FORMULATION

1. The Atmospheric Model.

As an approximation to the atmosphere in middle latitudes, a model of the undisturbed state is characterized as follows: (a) the undisturbed wind is zonal; the speed is constant and independent of the coordinates throughout the troposphere and stratosphere; (b) the lapse rate of temperature is constant in the troposphere and zero in the stratosphere; (c) both troposphere and stratosphere are barotropic, incompressible and frictionless layers.

2. Fundamental Equations.

It is assumed for purposes of mathematical simplicity that the curvature of the earth can be neglected. This assumption is valid when length of wave is small compared with circumference of the zonal circle along which the wave moves. This planar motion is expressed in a rectangular system of coordinates x, y and z with x increasing to the east, y northward, and z vertically upward. Then $\bar{u}$, $\bar{v}$, $\bar{w}$ denote corresponding total velocity components (consisting of undisturbed velocity and perturbation velocity). Further, denote total pressure by $\bar{p}$, angular velocity of the earth by Ω, acceleration of gravity (assumed constant) as g, geographical latitude by φ , the z and y components of Coriolis acceleration, f and β respectively. The density $\bar{\rho}$, having been assumed incompressible, is constant and

equal to the unperturbed value ρ . The Eulerian equations of motion in ordinary cartesian coordinates, fixed in the earth, become

$$\frac{d\bar{u}}{dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \bar{v} f - \beta \bar{w} , \tag{1}$$

$$\frac{d\bar{v}}{dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} - f \bar{u} , \tag{2}$$

$$\frac{d\bar{w}}{dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial z} + \beta \bar{u} - g . \tag{3}$$

A fourth equation is obtained from the assumption that the three-dimensional velocity divergence is zero.

$$\nabla_3 \cdot \mathbb{V} = 0 . \tag{4}$$

A relationship between density and height in the troposphere (with the assumption that g is constant with height) provides the fifth equation

$$\rho = \rho_0 \left[1 - \alpha z\right]^{\frac{g}{R\Gamma} - 1} . \tag{5}$$

But the density-height relationship in the stratosphere is

$$\rho = \rho_0 \left[1 - h_T \alpha\right]^{\frac{g}{R\Gamma} - 1} e^{-\frac{g}{RT_2}(z - h_T)} , \tag{6}$$

where h_T is height of tropopause.

3. The Boundary Conditions.

The boundary conditions express the following physical properties of the motion: (a) the perturbation components are continuous across the tropopause, that is,

$$\Delta\bar{\rho} = \Delta\bar{u} = \Delta\bar{v} = \Delta\bar{w} = \Delta\bar{p} = \Delta\bar{T} = 0. \tag{7}$$

(b) the momentum vanishes at the limit of the atmosphere.

$$\lim_{z\to\infty} \rho\bar{u} = \lim_{z\to\infty} \rho\bar{v} = \lim_{z\to\infty} \rho\bar{w} = 0. \tag{8}$$

4. Derivation of the Perturbation Equation.

The fundamental equations (1-6) together with the boundary conditions (7), (8) impose the necessary restrictions on the theoretical model. With the convention of a capital letter for steady state value and a small letter for the same quantity in the perturbed state, the characterization of the steady state is given by the conditions

$$V = W = \frac{dU}{dt} = \frac{\partial P}{\partial t} = \frac{\partial P}{\partial x} = 0 \quad , U = \text{constant.} \tag{9}$$

Equation (2) implies that the undisturbed flow must satisfy the condition of geostrophic equilibrium

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fU. \tag{10}$$

Equation (10) implies that ρ and P are functions of y and z only. Equation (3) expresses the condition for hydrostatic equilibrium in the mean state

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g, \tag{11}$$

with the omission of βU, which can be shown both empirically and theoretically to be negligible in comparison to the acceleration of gravity.

By means of the equation of state, and equations (4) and (5) with prescribed values of U, P_0 T_0, Γ, and h_T, the theoretical steady state model is completely described. The boundary conditions(7) and (8) are also satisfied by our steady state model. Hence the model prescribed is consistent with the fundamental equations (1-6) and the boundary conditions (7), (8).

5. The Linearized Equations.

The motion to be investigated can be treated as a small perturbation with velocity components u, v, and w superimposed on an undisturbed constant zonal current U, with the assumption of infinite lateral extent. Thus the motion is described by

$$\bar{u} = u(x,z,t) + U, \quad \bar{v} = v(x,z,t), \quad \bar{w} = w(x,z,t) \quad . \tag{12}$$

Similar expressions for the density and pressure in the disturbed state are

$$\bar{p} = p(x,z,t) + P(y,z), \quad \bar{\rho} = \rho(y,z) \, . \tag{13}$$

In deriving the perturbation equations the usual assumptions are made. (1) The total disturbed plus undisturbed motion satisfies the hydrodynamic equations, as well as the undisturbed motion alone. (2) The perturbations are so small that terms of second order in the

perturbation quantities can be neglected with respect to terms of the first order in the perturbation quantities. Linearizing the Eulerian equations of motion (1-3) and simplifying by means of the steady state equations (9-11), the system (1-3) becomes:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \beta w - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} , \tag{14}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + fu = 0 , \tag{15}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} - \beta u = -\frac{1}{\rho}\frac{\partial p}{\partial z} , \tag{16}$$

including the velocity divergence,

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 . \tag{17}$$

IV. THE BAROTROPIC PERTURBATION

1. The Form of the Perturbation and Resulting Conditions.

The perturbation components, traveling in the x- direction at a constant velocity, are assumed to be simple harmonic motions of infinite lateral extent. The amplitudes of the perturbation are included in a general expression of the z-coordinate, as follows

$$u = \sin m(x-ct)\, F(z); \quad v = \cos m(x-ct)\, Q(z);$$
$$w = -m \cos m(x-ct)\, H(z); \quad p = \sin m(x-ct)\, G(z). \tag{18}$$

To simplify the mathematical difficulties, the assumption is made that F(z) is proportional to Q(z), D being the constant of proportionality. Since u and w of equations (18) are related by the condition of zero velocity divergence, it can be shown that

$$F(z) = H'(z) = Q(z)\, D. \tag{19}$$

The equation for the velocity of the wave is derived by substituting the perturbation velocities (18) into the equation (15). With the equalities of (19) known, the wave velocity equation is

$$c = U - \frac{fD}{m} \tag{20}$$

Since U and m are real, c will be either real or complex, depending on D. The determination of D will be discussed fully in Chapter IV-2.

Substituting the perturbed velocity, pressure and mass fields into the perturbation equations (16) and (14) leads to

$$G'(z) = -\rho f D m H(z) + \rho\beta H'(z), \tag{21}$$

$$G(z) = \rho \left[\frac{-f(D^2-1)}{Dm}\right] H'(z) + \rho\beta H'(z), \tag{22}$$

respectively.

The differentiation of (22) with respect to z, leads to

$$G'(z) = \left[\frac{-f(D^2-1)}{Dm}\right] H''(z) + \frac{\partial\rho}{\partial z}\left[-\frac{f(D^2-1)}{Dm}\right] H'(z) + \rho\beta H'(z) + \frac{\partial\rho}{\partial z}\beta H(z). \tag{23}$$

By differentiation of density from equation (5), we have

$$\frac{\partial\rho}{\partial z} = -\rho \frac{\Gamma}{\Gamma_0}\left(\frac{g}{R\Gamma} - 1\right)(1-\alpha z)^{-1}. \tag{24}$$

Equating (21) and (23) and replacing ρ and $\frac{\partial\rho}{\partial z}$ by their expressions in (5) and (24) leads to the following ordinary differential equation for the amplitude function H(z):

$$(1-\alpha z)H''(z) - K H'(z) + \left[\frac{\beta K D m}{f(D^2-1} - \frac{D^2 m^2}{(D^2-1)}(1-\alpha z)\right] H(z) = 0 \tag{25}$$

where K is an abridged notation for

$$K = \alpha\,(g/R\Gamma - 1). \tag{26}$$

Using the further abridged notations,

$$A = \frac{\beta K D m}{f(D^2-1}, \qquad B = -\frac{D^2 m^2}{D^2-1}, \tag{27}$$

equation (25), the tropospheric differential equation becomes

$$(1-\alpha z)H''(z) - K H'(z) + [A + B(1-\alpha z)] H(z) = 0 . \qquad (28)$$

2. Perturbation Functions in the Stratosphere.

In exactly the same manner as described in the preceding section, the stratospheric differential equation follows by using (6) instead of (5) for the density expression. Thus, analogously to equation (24), the stratospheric density derivative is

$$\frac{\partial \rho}{\partial z} = -\rho / h_2 \;, \qquad (29)$$

where $h_2 = RT_2/g$ is the height of a homogeneous layer at stratospheric temperature T_2. We, therefore, obtain the second order linear differential equation with constant coefficients

$$H'(z) - \frac{H'(z)}{h_2} + \left[\frac{\beta D m}{f h_2 (D^2-1)} - \frac{D^2 m^2}{(D^2-1)}\right] H(z) = 0. \qquad (30)$$

The auxiliary equation of (30) is

$$0 = q^2 - \frac{q}{h_2} + \frac{\beta D m}{h_2 f (D^2-1)} - \frac{D^2 m^2}{D^2-1} \;, \qquad (31)$$

whence

$$q = \frac{1}{2h_2} \pm \frac{1}{2h_2}\sqrt{1 - \frac{4\beta D m h_2}{f(D^2-1)} \quad \frac{(2Dmh_2)^2}{(D^2-1)}} . \qquad (32)$$

The determination of the value of D is dependent upon the value assigned to the discriminant of the auxiliary equation (32). By analogy with the Rossby wave velocity

$$U - c = \beta / E m^2$$

it would follow that, for Rossby waves,

$$D_R = \beta/Efm \cong 0.14 .$$

The only restriction placed on D at this stage of the investigation *
is that $D < 1$, from which it follows at once, that the discriminant of (28) has a value n_2, at latitude 45°, slightly in excess of unity, $n_2 > 1$.

Solving the discriminant for D:

$$D^2\left[(2mh_2)^2-(n_2^2-1)\right]-(2mh_2)2D + (n_2^2-1) = 0, \tag{33}$$

let

$$Z^2 = (n_2^2-1)Y_2^2 = (2mh_2)^2. \tag{34}$$

Solving equation (33) by the quadratic formula gives

$$D = \frac{Y_2}{(n^2-1)^{\frac{1}{2}}} \frac{\pm Y_2}{(n^2-1)^{\frac{1}{2}}}\left[1-\frac{(Y_2^2-1)(n_2^2-1)}{Y_2^2}\right]^{\frac{1}{2}} \times \left[Y_2^2-1\right]^{-1}. \tag{35}$$

In order to expand the radical in (35) by means of the binomial expansion, the condition that

$$\left|Y_2^2-1\right| < Y_2^2/(n_2^2-1) = 2mh_2/(n_2^2-1)^2 \tag{36}$$

must hold.

* $D > 1$ with a moderately long wave of 6000 km, would give a value of $U-c > 100$ mps, so that obviously $D > 1$ does not apply.

Examining the two possible cases: (a) $Y_2 < 1$, (b) $Y_2 > 1$.

Case (a): $Y_2 < 1$.

Equation (36) becomes

$$(2mh_2)^2 > (n_2^2 - 1)/n_2^2 ,$$

so that

$$n_2^2 < 1 + 2mh_2 \quad , \tag{36'}$$

For L = 6000 km, $n_2^2 < 1.012$. Then from the inequality of case (a), the expansion of the radical of (35), with the choice of the negative sign, yields

$$D = \frac{(n_2^2-1)^{\frac{1}{2}}}{2Y_2} + \frac{1}{8}\,\frac{(Y_2^2-1)(n_2^2-1)^{3/2}}{Y_2^3} + \cdots \quad . \tag{37}$$

Case (b): $Y_2 > 1$

Equation (34) becomes

$$(2mh_2)^2/(n_2^2-1) > 1 .$$

It follows that

$$1 < n_2 < \left[1 + (2mh_2)^2\right]^{\frac{1}{2}} , \tag{38}$$

where $n_2 < 1.0004$ for a wave length of 6000 km. The value of n_2 determined by case (b) is included in that of case (a), so that $Y_2 < 1$ will be used in the evaluation of D. Moreover, all terms of (37) beyond the second will be considered negligible. Thus at latitude 45° where

$$\beta = f,$$

$$D = \frac{1}{2}\left(\frac{n_2^2 - 1}{2mh_2}\right) - \frac{1}{8}\left[\left(\frac{n_2^2-1}{2mh_2}\right)^3\left(1 - \frac{\{2mh_2\}^2}{n_2^2-1}\right)\right]. \qquad (39)$$

The evaluation of D, which for a particular mean latitude and wave number remains constant, is dependent on the value of n_2 selected from the inequality (36'). In this investigation, it was assumed desirable to have the ratio of the absolute maximum velocity of the horizontal perturbation to that of the vertical perturbation as large as consistently possible. This ratio, hereafter called

$$M = \left|\frac{u_{max}}{w_{max}}\right| = S_2\frac{(1-n_2)}{m}, \qquad (39')$$

can be shown to be a function of n_2 and L. The value of D is compared in Table 1. for various values of n_2 and wave number N.

N	n_2	m in meters^{-1} x 10^6	$2m\,h_2$	Y_2	M	D
4	1.0056	0.888	0.0113	0.107	0.5	.376
5	1.0070	1.110	0.0141	0.119	0.5	.378
6	1.0084	1.332	0.0170	0.132	0.5	.376
7	1.0098	1.554	0.0198	0.141	0.5	.377
8	1.0112	1.776	0.0226	0.151	0.5	.377
9	1.0125	1.998	0.0254	0.161	0.5	.377

Constant parameters: $\varphi = 45°$, $T_2 = 218°$ A

Table 1.

Having determined the value of D qualitatively in terms of m, h_2 and n_2, the general solution of the stratospheric equation (30) is

$$H(z) = C_1 e^{S_2 z - S_2 n_2 z} + C_2 e^{S_2 z + S_2 n_2 z}, \tag{40}$$

with the abridged notation

$$S_2 = 1/2h_2 .$$

By application of the boundary condition (8), $\rho w \underset{z\to\infty}{\to} 0$, it follows that $H(z) \underset{z\to\infty}{\to} 0$ if and only if the coefficient C_2 is made zero.

The particular solution satisfying the boundary conditions is

$$H(z) = C_1 e^{S_2(1-n_2)z} . \tag{41}$$

In the determination of the pressure perturbation in the stratosphere, the amplitude function G(z) is rewritten from equations (22) and (6), in the abridged notation, as

$$G(z) = C_1 \rho_0 \left(S_2 f \frac{[1-D^2][1-n_2]}{Dm} + \beta \right) (1-\alpha h_T)^{\frac{g}{R\Gamma}-1} e^{-S_2(n_2+1)z + \frac{h_T}{h_2}} . \tag{42}$$

From the solution of (41), the perturbation components of the stratosphere are defined. In abridged notation, they are

$$\begin{aligned} u_2 &= C_1 S_2 (1-n_2) e^{S_2(1-n_2)z} \sin m(x-ct), \\ v_2 &= \frac{C_1}{D} S_2 (1-n_2) e^{S_2(1-n_2)z} \cos m(x-ct), \\ w_2 &= -m C_1 e^{S_2(1-n_2)z} \cos m(x-ct), \\ p_2 &= G(z) \sin m(x-ct). \end{aligned} \tag{43}$$

In order to solve (43) for C_1, suitable numerical values must be assigned to the constant parameters φ, h_T, T_2, w_{max}, N and n_2, ρ_0. The following values are selected: $\varphi = 45^\circ$, $h_T = 10$ km, $T_2 = 218^\circ A$, $w_{max} = 5.5$ cm sec^{-1}, N = 5, $n_2 = 1.007$, $\rho_0 = 0.001121$ gm cm^{-3}. With assignment of these quantities, numerical values may be derived for the parameters m, S_2, M, where $m = 1.11 \times 10^{-6}$ meters^{-1}, $S_2 = .0785$ km^{-1}, $M = \frac{1}{2}$. Equation (43) yields $C_1 = -5 \times 10^4$ m^2 sec^{-2}.

3. An Approximate Solution in the Troposphere.

To obtain the solution of the barotropic wave by approximating the value of $(1 - \alpha z)$ in the tropospheric equation, (28) let

$$(1 - \alpha z) = T_3 / T_0 \tag{44}$$

be a satisfactory approximation. Substituting this approximation into equation (25), the equation becomes a second order differential equation with constant coefficients, namely,

$$\frac{T_3}{T_0} H''(z) - K H'(z) + \left(A + B\frac{T_3}{T_0}\right) H(z) = 0. \tag{45}$$

$$\Gamma = (1-\epsilon)\, g/R, \quad 0 < \epsilon < 1 \quad . \tag{46}$$

Then, if $h_3 = RT_3/g$, equation (45) can then be shown to be

$$H''(z) - \frac{\epsilon}{h_3} H'(z) + \left(\frac{\beta E D m}{f h_3 [D^2-1]} - \frac{D^2 m^2}{D^2-1}\right) H(z) = 0. \tag{47}$$

The general solution of this equation is dependent upon the discriminant obtained from the auxiliary equation of equation (47). By letting the discriminant equal n_3 , where

$$n_3 = 1 - \frac{4/3\, h_3}{f\epsilon(D^2-1)} + \frac{4(D m h_3)^2}{\epsilon^2(D^2-1)} , \tag{48}$$

and introducing the abridged notation $S_3 = \epsilon/2 h_3$,

with D known from the stratospheric solution, n_3 is determined.

Then the general solution in the approximate case is

$$H(z) = e^{S_3 z}\left[C_2 e^{n_3 S_3 z} + C_3 e^{-n_3 S_3 z}\right]. \tag{49}$$

4. Application of the Tropopause Boundary Condition.

The method of evaluating the coefficients C_2 and C_3 is the same for both the approximate and actual case. The approximate case is now considered.

With the boundary condition of a discontinuity of the first order at the tropopause the perturbation components must be continuous across the tropopause,

$$u_2 = u_3 \ , \quad w_2 = w_3 \ , \quad v_2 = v_3, \ p_2 = p_3 \ . \tag{50}$$

Then from (49) the first derivative with respect to z is

$$H'(z) = S_3 e^{S_3 z}\left[C_2(1+n_3)e^{n_3 S_3 z} + C_3(1-n_3)e^{-n_3 S_3 z}\right]. \tag{51}$$

so that

$$
\begin{aligned}
w_3 &= -m e^{S_3 z}\left(C_2 e^{n_3 S_3 z} + C_3 e^{-n_3 S_3 z}\right) \cos m(x-ct), \\
u_3 &= S_3 e^{S_3 z}\left[C_2 (1+n_3) e^{n_3 S_3 z} + C_3 (1-n_3) e^{-n_3 S_3 z}\right] \sin m(x-ct), \\
w_{3(max)} &= -m e^{S_3 z}\left(C_2 e^{n_3 S_3 z} + C_3 e^{-n_3 S_3 z}\right), \\
u_{3(max)} &= S_3 e^{S_3 z}\left[C_2 (1+n_3) e^{n_3 S_3 z} + C_3 (1-n_3) e^{-n_3 S_3 z}\right].
\end{aligned}
\tag{52}
$$

From the boundary condition (50), equate vertical velocities and also horizontal velocities at the tropopause, thus

$$
\begin{aligned}
C_1 e^{S_2(1-n_2)h_T} &= e^{S_3 h_T}\left(C_2 e^{n_3 S_3 h_T} + C_3 e^{-n_3 S_3 h_T}\right), \\
C_1 S_2 (1-n_2) e^{S_2(1-n_2)h_T} &= S_3 e^{S_3 h_T}\left[C_2 (1+n_3) e^{n_3 S_3 h_T} + C_3 (1-n_3) e^{-n_3 S_3 h_T}\right].
\end{aligned}
\tag{53}
$$

By solving these two equations simultaneously C_2 and C_3 are determined.

5. The Complete Solution in the Approximate Case.

With C_2 and C_3 determined, the perturbations of the troposphere are by approximation defined as

$$
\begin{aligned}
u_3 &= u_{3(max)} \sin m(x-ct), \\
v_3 &= \frac{1}{D} u_{3(max)} \cos m(x-ct), \\
w_3 &= w_{3(max)} \cos m(x-ct), \\
p_3 &= \rho_0 (1-\alpha z)^{\frac{g}{R\Gamma}-1}\left[\frac{f(1-D^2)}{Dm} u_{3(max)} - \frac{\beta}{m} w_{3(max)}\right] \sin m(x-ct).
\end{aligned}
\tag{54}
$$

In order to solve (54), suitable numerical values must be assigned to the parameters of the troposphere, in addition to those assigned in the stratosphere (e.g., Section 2 of this Chapter). The additional parameters, having selected values, are

$T_3 = 256.5°A, \Gamma = 5.18° km^{-1}$. With assignment of these quantities it can be shown that the constants C_2 and C_3 have the values 3.59 and -4.98×10^4, respectively. A diagram of the perturbation velocities is presented in Figure 1. for a graphical representation of the solution determined in this Chapter.

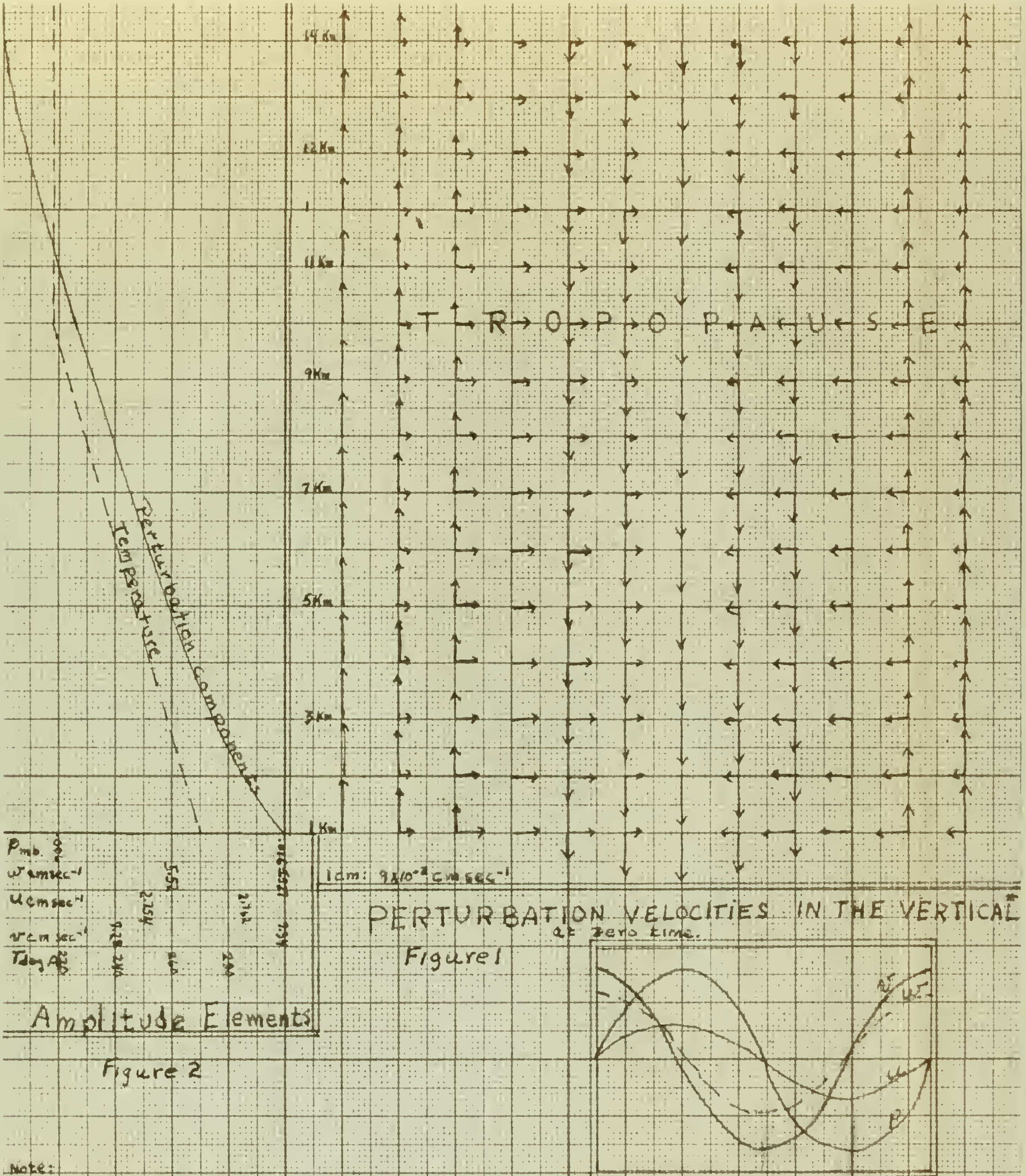

Figure 1

Figure 2

Note:
*To get vertical perturbation add 5.437 cm sec-1 to vertical vector shown
~ add 2.718 cm sec-1 to horizontal vector shown, to get u.

Phase Relationships: pressure - perturbation velocities
Figure 3

V. THE VARIABLE COEFFICIENT CASE

1. The Solution of the Perturbation.

With D determined and constant throughout the atmosphere for any particular mean latitude, the second order differential equation (28) is now solvable. By suitable manipulation, described in the Appendix, this equation (28) is a case of the confluent hypergeometric differential equation

$$\eta L''(\eta) + [b-\eta] L'(\eta) - a L(\eta) = 0,$$

$$\eta = \frac{KB}{\alpha A}\left[\frac{B}{A} + 1 - \alpha z\right]. \qquad (55)$$

Equation (55) is satisfied by the functions

$$\psi_1 = F(a, b, \eta) = 1 + \frac{a}{b}\eta + \frac{1}{2}\frac{a(a+1)}{b(b+1)}\eta^2 + \cdots ,$$

$$\psi_2 = (-\eta)^{1-b} F(a-b+1, 2-b, \eta) =$$

$$1 + \frac{a-b+1}{2-b}\eta + \frac{1}{2}\frac{(a-b+1)(a-b+2)}{(2-b)(3-b)}\eta^2 + \cdots \qquad (56)$$

The confluent hypergeometric equation then has the general solution

$$H(z) = C_2 \psi_1 + C_3 \psi_2 . \qquad (57)$$

To test the convergence of $\psi_1 = F(a, b, \eta)$, compare to the series e^x, where $x = -a/b\,|\eta|$. It can be shown that, termwise, $\psi_1 < e^x$ and orders higher than the second can be dropped with accuracy to the first five significant figures. Then

$$\psi_1 \cong 1 + \frac{a}{b}\eta + \frac{1}{2}\,\frac{a(a+1)}{b(b+1)}\eta^2 , \tag{58}$$

where $a/b = 1/D^2 - 1$.

Testing the convergence of

$$(-\eta)^{-(1-b)}\,\psi_1 = F[(a-b+1),(2-b),\eta] ,$$

after the first 12 terms, the series is more rapidly convergent than where $x = -a/b\,|\eta|$. In the range of values used in this study, however, there is negligible error, if series is shortened to

$$(-\eta)^{-(1-b)}\,\psi_2 = 1 + \frac{a-b+1}{2-b} + \frac{1}{2}\,\frac{(a-b+1)(a-b+2)}{(2-b)(3-b)}\eta^2 . \tag{59}$$

The general solution in the troposphere from equation (57) is

$$H(z) = C_2\left(1 + \frac{a}{b}\eta + \frac{1}{2}\,\frac{a(a+1)}{b(b+1)}\eta^2\right) + C_3(-\eta)^{1-b}\left[1 + \frac{a-b+1}{2-b}\eta + \frac{1}{2}\,\frac{(a-b+1)(a-b+2)}{(2-b)(3-b)}\eta^2\right] . \tag{60}$$

2. Application of the Tropopause Boundary Conditions.

After the manner of evaluating coefficients in Chapter IV, equating the vertical and horizontal velocity across the tropopause, two resultant equations can be solved simultaneously for C_2 and C_3.

3. The Complete Solution.

With C_2 and C_3 determined, the perturbations in the troposphere are defined as

$$u_2 = \left(C_2 \psi_2' + C_3 \left[\psi_2'(-\eta)^{1-b} + (-\eta)^{-b}\eta'(1-b)\psi_2\right]\right) \sin m(x-ct),$$

$$w_2 = -m\left(C_2 \psi_1 + C_3 \psi_2\right) \cos m(x-ct),$$

$$v_2 = \left[\frac{C_2}{D}\psi_1' + \frac{C_3}{D}\left(\{-\eta\}^{1-b}\psi_2' + \{-\eta\}^{-b}\eta'\{1-b\}\psi_2\right)\right] \cos m(x-ct), \tag{61}$$

$$p_2 = p_0 [1-\alpha z]^{\frac{g}{R\Gamma}-1} \left[\frac{f(1-D^2)}{Dm}\left(C_2 \psi_1' + C_3\left\{(-\eta)^{1-b}\psi_2' + (-\eta)^{-b}\eta'(1-b)\psi_2\right\}\right) + \beta\left(C_2\psi_1 + C_3\psi_2\right)\right] \sin m(x-ct),$$

$$\eta = \frac{Dm}{\alpha}\left(\frac{Dm}{K} - [1-\alpha z]\right).$$

In (61) the primes denote differentiation with respect to z. The values of parameters to be inserted in (61) are

$D = .377$, $m = 1.1 \times 10^{-8}\ cm^{-1}$, $\alpha = 1.9 \times 10^{-7}\ cm^{-1}$,

$K = 1.064 \times 10^{-6}\ cm^{-1}$, so that η can be evaluated for any value of z. In particular, at the tropopause η has the value $\eta = -.0176$. Likewise $\eta' = .415 \times 10^{-8}$, and is constant throughout the troposphere.

The complete solution includes the solution above the tropopause which remains precisely as developed in Chapter IV-2.

VI. DISCUSSION OF RESULTS

It would appear that the variations of the value of n_2 determined by the discriminant of the stratospheric auxiliary equation (32) plays the most important role in this entire investigation. By variation of this parameter alone, the wave velocity, the stationary wave length and the ratio of horizontal to the vertical maximum velocity, can be varied. Within the allowable range of this parameter, wave velocities determined by Rossby are included.

Gravity and sound waves are also included in these solutions for appropriate values of n. The values of n correspond to the inequality $n^2 - 1 < 2 m h_2 = 4\pi h_2 / L$, with values of the wave-length, L, suitably chosen to be representative values of wave-length for sound and gravity waves, respectively. It will be remembered that Charney [5], in dealing with the problem of the unstable baroclinic wave, first eliminated these short stable waves, which in the present paper is equivalent to restricting the condition to D < 1.

A further possible investigation is suggested by this study. By limiting the lateral extent of the disturbance, larger values of the velocity components are consistent with the small values of vertical divergence used in this study, the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ would then partially compensate one another, giving the proper (small) magnitude to $\nabla_H \cdot V = -\frac{\partial w}{\partial z}$.

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APPENDIX I.

The transformation of equation (28) to the form of a confluent hypergeometric differential equation follows.

Equation (28) is restated here.

$$(1-\alpha z)H''(z) - K H'(z) + \left[A + B(1-\alpha z)\right] H(z) = 0. \qquad (28)$$

Divide by (A+B):

$$\left(\frac{1-\alpha z}{A+B}\right) H''(z) - \frac{K}{A+B} H'(z) + \left(1 - \frac{B\alpha z}{A+B}\right) H(z) = 0. \qquad (28.a)$$

Divide by $\left(1 - B\alpha z \middle/ A+B\right) \neq 0,$ then multiply by (A+B)

$$\left(\frac{1-\alpha z}{1 - \frac{B\alpha z}{A+B}}\right) H''(z) - \left(\frac{K}{1 - \frac{B\alpha z}{A+B}}\right) H'(z) + (A+B) H(z) = 0. \qquad (28.b)$$

Momentarily, consider the first term only

$$\left(\frac{1-\alpha z}{1 - \frac{B\alpha z}{A+B}}\right) = \left[\frac{\frac{A+B}{B} - \left(\frac{B\alpha z}{A+B}\right)\left(\frac{A+B}{B}\right) - \frac{A+B}{B} + 1}{\left(1 - \frac{B\alpha z}{A+B}\right)}\right] =$$

$$\left[\frac{A+B}{B} - \frac{A}{B\left(1 - \frac{B\alpha z}{A+B}\right)}\right]. \qquad (28.c)$$

Expanding,

$$\left[1-\frac{B\alpha z}{A+B}\right]^{-1}=\left[1+\frac{B\alpha z}{A+B}\right], \tag{28.d}$$

if it is assumed that higher order terms negligible.

Equation (28.b) together with (28.c,d.) yield

$$\left[\frac{A+B}{B}+\frac{A}{B}\left(1+\frac{B\alpha z}{A+B}\right)\right]H''(z)-K\left[1+\frac{B\alpha z}{A+B}\right]H'(z)+\left[A+B\right]H(z)=0. \tag{28.e}$$

Collecting terms, (28.e) is

$$\left[1-\frac{A\alpha z}{A+B}\right]H''(z)-K\left[1+\frac{B\alpha z}{A+B}\right]H'(z)+\left[A+B\right]H(z)=0. \tag{28.f}$$

Let

$$\xi=1-\frac{A\alpha z}{A+B},\quad \frac{\alpha z}{A+B}=\frac{1-\xi}{A},\quad \frac{d\xi}{dz}=\frac{-A\alpha}{A+B},$$

then

$$1+\frac{B\alpha z}{A+B}=1+\frac{B}{A}-\frac{B\xi}{A},$$

and equation (28.f) is

$$\xi H''(z)-K\left(1+\frac{B}{A}\xi\right)H'(z)+(A+B)H(z)=0, \tag{28.g}$$

where

$$H(z) = H\left(\frac{A+B}{A\alpha}[1-\xi]\right) = J(\xi),$$

$$H'(z) = \frac{dJ}{d\xi}\frac{d\xi}{dz} = -J'(\xi)\frac{A\alpha}{A+B}$$

$$H''(z) = \frac{d^2J}{d\xi^2}\left(\frac{A\alpha}{A+B}\right)^2 = J''(\xi)\left(\frac{A\alpha}{A+B}\right)^2$$

Equation (28.g), in terms of ξ , becomes

$$\left(\frac{A\alpha}{A+B}\right)^2 \xi J''(\xi) + \frac{KA\alpha}{A+B}\left(1+\frac{B}{A}-\frac{B}{A}\xi\right)J'(\xi) + (A+B)J(\xi) = 0. \tag{28.h}$$

Dividing (28.h) by $\left[A\alpha/A+B\right]^2$ yields

$$\xi J''(\xi) + \left(\frac{K\alpha AB}{A[A+B]}\right)\left(\frac{A+B}{\alpha A}\right)\left(\frac{A+B}{B}-\xi\right)J'(\xi) + \frac{(A+B)^3}{(A\alpha)^2}J(\xi) = 0. \tag{28.i}$$

By letting $r = KB(A+B)/A^2\alpha$,

equation (28.i) becomes

$$\xi J''(\xi) + r\left(\frac{A+B}{B} - \xi\right)J'(\xi) + \frac{(A+B)^3}{(\alpha A)^2}J(\xi) = 0. \tag{28.j}$$

Then if $r\xi = \eta$,

$$J(\xi) = J\left(\frac{\eta}{r}\right) = L(\eta),$$

$$J'(\xi) = \frac{dL}{d\eta}\frac{d\eta}{d\xi} = r\,L'(\eta),$$

$$J''(\xi) = r\frac{d}{d\xi}\left(\frac{dL}{d\eta}\right) = r^2 L''(\eta),$$

equation (28.j) is

$$\xi r^2 L''(\eta) + r^2\left(\frac{A+B}{B} - \xi\right)L'(\eta) + \frac{(A+B)^3}{(\alpha A)^2}L(\eta) = 0. \qquad (28.k)$$

Divide equation (28.k) by r and substitute the equality $\eta = r\xi$, and the equation becomes

$$\eta L''(\eta) + \left[r\frac{(A+B)}{B} - \eta\right]L'(\eta) + \frac{(A+B)^3}{r(\alpha A)^2}L(\eta) = 0. \qquad (28.1)$$

If $b = r(A+B)/B$, $a = -(A+B)^3/r(\alpha A)^2$,

equation (28.1) is identically equation (55), which is the form of the confluent hypergeometric differential equation.